Stability of sampled systems

Digital control

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Stability of discrete-time systems

$$\frac{Y(z)}{R(z)} = \frac{G(z)}{1 + GH(z)} = \frac{N(z)}{D(z)},$$

where $1 + GH(z) = 0$ is also known as the characteristic equation

The stability of the system depends on the location of the poles of the closed-loop transfer function, or the roots of the characteristic equation $D(z) = 0$. left-hand side of the S-plane

Thus, we can say that a system in the z-plane will be stable if all the roots of the characteristic equation, $D(z) = 0$, lie inside the unit circle.

Stability of discrete-time systems

There are several methods available to check for the stability of a discrete-time system:

- Factorize $D(z) = 0$ and find the positions of its roots, and hence the position of the closed-loop poles.
- Determine the system stability without finding the poles of the closed-loop system, such as Jury's test.
- Transform the problem into the s-plane and analyse the system stability using the well-established s-plane techniques, such as frequency response analysis or the Routh–Hurwitz criterion.
- Use the root-locus graphical technique in the z-plane to determine the positions of the system poles.

Factorizing characteristic equation: example

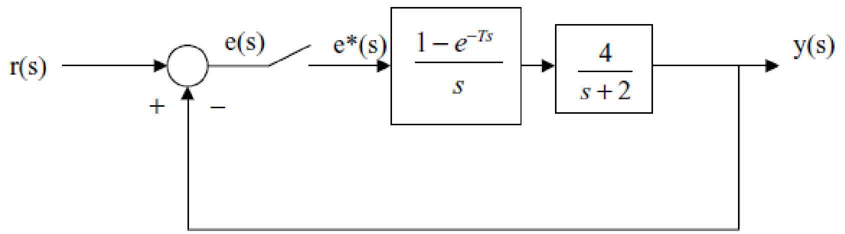

Figure 8.1 Closed-loop system

Example 8.1

The block diagram of a closed-loop system is shown in Figure 8.1. Determine whether or not the system is stable. Assume that $T = 1$ s.

Factorizing characteristic equation: example

Solution

The closed-loop system transfer function is

$$\frac{Y(z)}{R(z)} = \frac{G(z)}{1 + G(z)}, \tag{8.1}$$

where

$$G(z) = Z\left\{\left[\frac{1 - e^{-Ts}}{s}\frac{4}{s+2}\right]\right\} = (1 - z^{-1})Z\left\{\left[\frac{4}{s(s+2)}\right]\right\} = (1 - z^{-1})\frac{2z(1 - e^{-2T})}{(z-1)(z - e^{-2T})}$$

$$= \frac{2(1 - e^{-2T})}{z - e^{-2T}}. \tag{8.2}$$

For $T = 1$ s,

$$G(z) = \frac{1.729}{z - 0.135}.$$

The roots of the characteristic equation are $1 + G(z) = 0$, or $1 + 1.729/(z - 0.135) = 0$, the solution of which is $z = -1.594$ which is outside the unit circle, i.e. the system is not stable.

Factorizing characteristic equation: example

Example 8.2

For the system given in Example 8.1, find the value of T for which the system is stable.

Factorizing characteristic equation: example

Solution

From (8.2),

$$G(z) = \frac{2(1 - e^{-2T})}{z - e^{-2T}}.$$

The roots of the characteristic equation are $1 + G(z) = 0$, or $1 + 2(1 - e^{-2T})/(z - e^{-2T}) = 0$, giving

$$z - e^{-2T} + 2(1 - e^{-2T}) = 0$$

Factorizing characteristic equation: example

or

$$z = 3e^{-2T} - 2.$$

The system will be stable if the absolute value of the root is inside the unit circle, i.e.

$$|3e^{-2T} - 2| < 1,$$

from which we get

$$2T < \ln\left(\tfrac{1}{3}\right) \quad \text{or} \quad T < 0.549.$$

Thus, the system will be stable as long as the sampling time $T < 0.549$.

Jury's test

Jury's stability test is similar to the Routh–Hurwitz stability criterion used for continuous-time systems. Although Jury's test can be applied to characteristic equations of any order, its complexity increases for high-order systems.

To describe Jury's test, express the characteristic equation of a discrete-time system of order n as

$$F(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0 = 0, \tag{8.3}$$

where $a_n > 0$. We now form the array shown in Table 8.1. The elements of this array are defined as follows:

- The elements of each of the even-numbered rows are the elements of the preceding row, in reverse order.
- The elements of the odd-numbered rows are defined as:

$$b_k = \begin{vmatrix} a_0 & a_{n-k} \\ a_n & a_k \end{vmatrix}, \quad c_k = \begin{vmatrix} b_0 & b_{n-k-1} \\ n_{n-1} & b_k \end{vmatrix}, \quad d_k = \begin{vmatrix} c_0 & c_{n-2-k} \\ c_{n-2} & c_k \end{vmatrix}, \quad \ldots.$$

Jury's test

Table 8.1 Array for Jury's stability tests

z^0	z^1	z^2	$\dots$	z^{n-k}	$\dots$	z^{n-1}	z^n
a_0	a_1	a_2	$\dots$	a_{n-k}	$\dots$	a_{n-1}	a_n
a_n	a_{n-1}	a_{n-2}	$\dots$	a_k	$\dots$	a_1	a_0
b_0	b_1	b_2	$\dots$	b_{n-k}	$\dots$	b_{n-1}	
b_{n-1}	b_{n-2}	b_{n-3}	$\dots$	b_{k-1}	$\dots$	b_0	
c_0	c_1	c_2	$\dots$	c_{n-k}	$\dots$		
c_{n-2}	c_{n-3}	c_{n-4}	$\dots$	c_{k-2}	$\dots$		
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$			
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$			
l_0	l_1	l_2	l_3				
l_3	l_2	l_1	l_0				
m_0	m_1	m_2					

Jury's test

The necessary and sufficient conditions for the characteristic equation (8.3) to have roots inside the unit circle are given as

$$F(1) > 0, \quad (-1)^n F(-1) > 0, \quad |a_0| < a_n, \tag{8.4}$$

$$\begin{aligned} |b_0| &> b_{n-1} \\ |c_0| &> c_{n-2} \\ |d_0| &> d_{n-3} \\ &\cdots \\ &\cdots \\ |m_0| &> m_2. \end{aligned} \tag{8.5}$$

Jury's test is then applied as follows:

- Check the three conditions given in (8.4) and stop if any of these conditions is not satisfied.
- Construct the array given in Table 8.1 and check the conditions given in (8.5). Stop if any condition is not satisfied.

Jury's test

Jury's test can become complex as the order of the system increases. For systems of order 2 and 3 the test reduces to the following simple rules. Given the second-order system characteristic equation

$$F(z) = a_2 z^2 + a_1 z + a_0 = 0, \quad \text{where } a_2 > 0,$$

no roots of the system characteristic equation will be on or outside the unit circle provided that

$$F(1) > 0, \quad F(-1) > 0, \quad |a_0| < a_2..$$

Given the third-order system characteristic equation

$$F(z) = a_3 z^3 + a_2 z^2 + a_1 z + a_0 = 0, \quad \text{where } a_3 > 0,$$

no roots of the system characteristic equation will be on or outside the unit circle provided that

$$F(1) > 0, \quad F(-1) < 0, \quad |a_0| < a_3,$$

$$\left|\det\begin{bmatrix} a_0 & a_3 \\ a_3 & a_0 \end{bmatrix}\right| > \left|\det\begin{bmatrix} a_0 & a_1 \\ a_3 & a_2 \end{bmatrix}\right|.$$

Jury's test

Example 8.3

The closed-loop transfer function of a system is given by

$$\frac{G(z)}{1+G(z)},$$

where

$$G(z) = \frac{0.2z + 0.5}{z^2 - 1.2z + 0.2}.$$

Determine the stability of this system using Jury's test.

Jury’s test

Solution

The characteristic equation is

$$1 + G(z) = 1 + \frac{0.2z + 0.5}{z^2 - 1.2z + 0.2} = 0$$

or

$$z^2 - z + 0.7 = 0.$$

Applying Jury’s test,

$$F(1) = 0.7 > 0, \quad F(-1) = 2.7 > 0, \quad 0.7 < 1.$$

All the conditions are satisfied and the system is stable.

Jury’s test

Example 8.4

The characteristic equation of a system is given by

$$1 + G(z) = 1 + \frac{K(0.2z + 0.5)}{z^2 - 1.2z + 0.2} = 0.$$

Determine the value of K for which the system is stable.

Solution

The characteristic equation is

$$z^2 + z(0.2K - 1.2) + 0.5K = 0, \quad \text{where } K > 0.$$

Applying Jurys’s test,

$$F(1) = 0.7K - 0.2 > 0, \quad F(-1) = 0.3K + 2.2 > 0, \quad 0.5K < 1.$$

Thus, the system is stable for $0.285 < K < 2$.

Jury's test

Example 8.5

The characteristic equation of a system is given by

$$F(z) = z^3 - 2z^2 + 1.4z - 0.1 = 0.$$

Determine the stability of the system.

Solution

Applying Jury's test, $a_3 = 1$, $a_2 = -2$, $a_1 = 1.4$, $a_0 = -0.1$ and

$$F(1) = 0.3 > 0, \quad F(-1) = -4.5 < 0, \quad 0.1 < 1.$$

The first conditions are satisfied. Applying the other condition,

$$\left|\begin{bmatrix} -0.1 & 1 \\ 1 & -0.1 \end{bmatrix}\right| = -0.99 \quad \text{and} \quad \left|\begin{bmatrix} -0.1 & 1.4 \\ 1 & -2 \end{bmatrix}\right| = -1.2;$$

since $|0.99| < |-1.2|$, the system is not stable.